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PROPOSED LETTER TO THE EDITOR

EVALUATION OF A DEFINITE INTEGRAL COMMONLY FOUND IN MAGNETIC FIELD PROBLEMS BY THE USE OF LENGENDRE POLYNOMIALS

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EVALUATION OF A DEFINITE INTEGRAL COMMONLY FOUND IN MAGNETIC FIELD PROBLEMS BY THE USE OF LEGENDRE POLYNOMIALS

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I INTRODUCTION

In calculating magnetic fields of configurations that possess axial symmetry, such as solenoids and cones, a definite integral is encountered that has not yet been placed in tabulated form. The integral in question is

$$I(r) = \int_0^\pi \cos\omega \cos\theta \ln\left(R - r_0\cos\omega + \sqrt{R^2 + r^2 - 2Rr\cos\omega}\right)d\theta \tag{1}$$

where

$$\cos\omega = \cos\alpha\cos\varphi + \sin\alpha\sin\varphi\cos\theta$$

This type of integral was calculated previously by use of numerical integration by Brown, Flax, Itean, and Laurence [1], where it was written in terms of cylindrical coordinates instead of the form presented in equation (1), which is in spherical coordinates. This type of integral appears when the magnetic field intensity is calculated on or off the axes of cylindrical current sheets or finite thick solenoids.

A representation of the integral in equation (1) in closed form, in terms of Legendre polynomials, is obtained. This is desirable for the purpose of numerical calculation since the evaluation of the polynomials can be performed by the use of recurrence formulae.

II EVALUATING THE DEFINITE INTEGRAL

Differentiation of equation (1) with respect to r, which is not a variable of integration, yields

$$I'(r) = \int_0^{\pi} \frac{\cos\omega\cos\theta(-V\cos\omega + r\sin\omega - u\cos\omega)d\theta}{(U+V)V} \tag{2}$$

where

$$U = R - r\cos\omega$$

$$V = \sqrt{R^2 + r^2 - 2Rr\cos\omega}$$

On multiplying the integrand by $\frac{U-V}{U-V}$ and rearranging terms, we obtain

$$I'(r) = -\int_0^{\pi} \frac{R\cos\omega\cos\theta\, d\theta}{r(R^2 + r^2 - 2Rr\cos\omega)^{1/2}} + \int_0^{\pi} \frac{\cos\omega\cos\theta}{r} \tag{3}$$

Equation (3) can be put into suitable Legendre form:

$$I'(r) = -\int_0^{\pi} \frac{1}{r}\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n P_n(\cos\omega)\cos\omega\cos\theta\, d\theta + \int_0^{\pi} \frac{\cos\omega\cos\theta}{r}\, d\theta$$

$$I'(r) = -\int_0^{\pi} \frac{1}{r}\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n P_n(\cos\omega)\cos\omega\cos\theta\, d\theta \qquad \text{for } R > r \tag{4}$$

For $r > R$, the same procedure is followed. We find that $I'(r)$ is of the same form as that previously found with r and R interchanged.

Using the addition theorem for Legendre polynomials, namely

$$P_n(\cos\omega) = P_n(\cos\alpha)\,P\;(\cos\varphi)$$

$$+ L \sum_{m=1}^{n} \frac{(n - m)!}{(n + m)!}\, P\;(\cos\alpha) P\;(\cos\varphi) \cos m\varphi \qquad (5)$$

and integrating with respect to θ, we obtain

$$I'(r) = -\pi \cos\alpha \cos\varphi \sum_{n=1}^{\infty} \frac{r^{n-1}}{R^n} \frac{1}{n(n + 1)} P_n^1(\cos\alpha) P_n^1(\cos\varphi)$$

$$-2\pi \sin\alpha \sin\varphi \sum_{n=1}^{\infty} \frac{(r)^{n-1}}{R^n} P_n(\cos\alpha) P_n(\cos\varphi)$$

$$-\frac{\pi}{2} \sin\alpha \sin\varphi \sum_{n=2}^{\infty} \frac{(n - 2)!}{(n + 2)!} \frac{r^{n-1}}{R^n} P_n^2(\cos\alpha) P_n^2(\cos\varphi) \qquad (6)$$

Integrating with respect to r, in order to negate the differentiation leading to equation (2), and evaluating the constant of integration by suitable boundary conditions, reduces equation (6) to

$$I(r) = -\pi \cos\alpha \cos\varphi \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \frac{1}{n^2(n + 1)} P_n^1(\cos\alpha) P_n^1(\cos\varphi)$$

$$-\frac{\pi}{2} \sin\alpha \sin\varphi \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \frac{1}{n} P_n(\cos\alpha) P_n(\cos\varphi)$$

$$-\frac{\pi}{2} \sin\alpha \sin\varphi \sum_{n=2}^{\infty} \frac{1}{n} \frac{(n - 2)!}{(n + 2)!} \left(\frac{r}{R}\right)^n P_n^2(\cos\alpha) P_n^2(\cos\varphi)$$

$$-\frac{\pi}{2} \sin\alpha \sin\varphi \ln 2R \qquad (7)$$

III CONCLUSIONS

A

The use of Legendre polynomials in calculating magnetic fields has been demonstrated by Garrett [2] and Flax and Callaghan [3]. The major advantage of Legendre polynomials is that they can be used to obtain closed-form solutions in terms of tabulated functions. These solutions have been programmed using recurrence relations.

REFERENCES

1. Brown, Gerald V., Flax, Lawrence, Itean, Eugene C., and Laurence, James C.: Axial and Radial Magnetic Fields of Thick, Finite-Length Solenoids. NASA TR R-170, 1963.

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